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TOPOGRAPHIC EFFECTS ON SOLITARY ROSSBY WAVES

PIERRE HENROTAY

Mécanique des Fluides Géophysiques, Université de Liège, Liège (Belgium)

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ABSTRACT

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The model for baroclinic solitary waves previously described by Flierl is developed to include effects of bottom variations.

Topographic disturbances are shown to act, through their first and second derivatives, on both the phase speed correction and the maximum horizontal amplitude.

The combined effects of mean shear flow steepening and bottom variations are considered; an equation is derived, showing the relative importance of each effect.

1. INTRODUCTION

Recent studies in planetary waves dynamics (e.g., Larichev and Reznik, 1976; Redekopp, 1977; Clarke, 1971; Flierl, 1979; Malanotte Rizzoli, 1980; Malanotte Rizzoli and Hendershott, 1980) have shown the possibility of the existence of isolated disturbances ('solitary waves'), both in the atmosphere and ocean.

These solitary solutions of the quasigeostrophic equations appear to have some exciting characteristics, particularly suited to the description of mesoscale eddies or frontal eddies like Gulf Stream rings.

One of the formative studies of solitary disturbances is that of Flierl (1979): his model uses the β-plane equations in the continuously stratified form and reveals the existence of radially symmetric, baroclinic solutions, which is most desirable for applications to oceanic mesoscale eddies and synoptic weather systems. It requires a mean shear flow far from the isolated wave: the eddy is steepened by its interaction with the shear; the potential vorticity—stream function relationship is continuous.

Nevertheless, Flierl's model (rigid lid and flat bottom assumptions) neglects topographic effects, which can support solitary waves (Clarke, 1971; Malanotte Rizzoli and Hendershott, 1980). The joint effect of curved topography and interaction with a mean flow is worth investigating. The purpose here is to extend Flierl's results in order to include effects of bottom variations.

In Section 2, we recall the set of equations previously obtained and note the change in the boundary conditions introduced by our varying bottom hypothesis. In Section 3, we consider the case of pure topographic effects (no mean flow) and discover that solitary disturbances with radial symmetry are possible. The dependence of the phase speed on the bottom slope is shown. In Section 4, the influence of an arbitrary mean flow is considered. In Section 5, a solution corresponding to the case of a large scale mean flow is deduced, and compared to Flierl's result. A general equation is obtained, which shows the relative effects of the mean flow and of the depth variation. An example is provided, which shows that the two effects may be cooperative or destructive.

2. GOVERNING EQUATIONS

The quasigeostrophic, β-plane equations in the continuously stratified form can be written as follows (conservation of quasigeostrophic potential vorticity)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\nabla^2\psi + f_0 + \beta y + \frac{\partial}{\partial z}\frac{f_0^2}{N^2}\frac{\partial}{\partial z}\psi\right\} = 0$$

where

$\psi(x, y, z, t)$ is the stream function

$x(u)$ is the eastward distance (velocity)

$y(v)$ is the northward distance (velocity)

$z(\omega)$ is the upward distance (velocity)

$N^2(z)$ is the specified Brunt—Väisälä frequency

$f_0 + \beta y$ is the Coriolis parameter with N—S variation

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} = \frac{\partial}{\partial t} + J(\psi,)$$

∇^2 is the horizontal Laplacian operator

The fluid is contained by a rigid horizontal upper surface and by an irregular lower surface. The boundary conditions are (Flierl, 1978)

$\omega = 0$ at $z = 0$

$\omega = J(\psi, b)$ at $z = -H$

where H is the mean depth, and $b(x, y)$ is the bottom deviation (the true bottom lies at $z = -H + b$). The conditions can also be written

$$\left(\omega = -\frac{f_0}{N^2}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\psi}{\partial z}\right)$$

$$\frac{d}{dt}\frac{\partial \psi}{\partial z} = 0 \qquad \text{at } z = 0$$

$$\frac{\partial}{\partial t}\frac{\partial \psi}{\partial z} + J\left(\psi, \frac{\partial \psi}{\partial z} + \frac{N^2 b}{f_0}\right) = 0 \qquad \text{at } z = -H$$

Following Flierl (1979), we rewrite these equations for motions which translate steadily in the x-direction. We only allow undisturbed zonal mean flows $U(y, z)$, so we have

$$\frac{\partial b}{\partial x} = 0$$

$$\psi(x, y, z, t) = \chi(x - ct, y, z)$$

Substitution in previously obtained equations leads to

$$J\left(\chi + cy, \nabla^2\chi + \beta y + \frac{\partial}{\partial z}\frac{f_0^2}{N^2}\frac{\partial}{\partial z}\chi\right) = 0$$

$$J\left(\chi + cy, \frac{\partial \chi}{\partial z}\right) = 0 \qquad \text{at } z = 0$$

$$J\left(\chi + cy, \frac{\partial \chi}{\partial z} + \frac{N^2}{f_0} b\right) = 0 \qquad \text{at } z = -H$$

Integrating these equations by introducing the potential vorticity functional $P(Z, z)$ and surface functionals $G_0(Z)$ and $G_1(Z)$, the following set of equations is obtained (only the third equation differs from Flierl's eq. 2.1)

$$\nabla^2\chi + \frac{\partial}{\partial z}\frac{f_0^2}{N^2}\frac{\partial}{\partial z}\chi + \beta y = P(\chi + cy, z)$$

$$\frac{\partial \chi}{\partial z} = G_0(\chi + cy) \qquad \text{at } z = 0 \qquad (2.1)$$

$$\frac{\partial \chi}{\partial z} = -\frac{N^2}{f_0} b(y) + G_1(\chi + cy) \qquad \text{at } z = -H$$

We look for the evolution of a perturbation superimposed on a zonal mean flow ψ

$$\chi = \Psi + \varphi, \qquad \frac{\partial \Psi}{\partial y} = -U(y, z)$$

The mean flow satisfies

$$\frac{\partial^2\Psi}{\partial y^2} + \frac{\partial}{\partial z}\frac{f_0^2}{N^2}\frac{\partial \Psi}{\partial z} + \beta y = P(\Psi + cy, z)$$

$$\frac{\partial \Psi}{\partial z} = G_0(\Psi + cy) \qquad \text{at } z = 0 \tag{2.2}$$

$$\frac{\partial \Psi}{\partial z} = -\frac{N^2}{f_0} b + G_1(\Psi + cy) \qquad \text{at } z = -H$$

while the perturbation φ is described by

$$\nabla^2 \varphi + \frac{\partial}{\partial z} \frac{f_0^2}{N^2} \frac{\partial \varphi}{\partial z} = P(\varphi + \Psi + cy, z) - P(\Psi + cy, z)$$

$$\frac{\partial \varphi}{\partial z} = G_0(\varphi + \Psi + cy) - G_0(\Psi + cy) \qquad \text{at } z = 0 \tag{2.3}$$

$$\frac{\partial \varphi}{\partial z} = G_1(\varphi + \Psi + cy) - G_1(\Psi + cy) \qquad \text{at } z = -H$$

We require (isolated disturbance)

$$\varphi \to 0 \qquad \text{as } |x| \to +\infty$$

The nonlinear PDS (2.3) is identical to that obtained by Flierl; the effect of the bottom variations appears implicitly through the functional G_1.

The knowledge of the mean flow Ψ and of b allows us to construct the functionals P, G_0 and G_1. The nonlinear PDS (2.3) can then be solved; it is in fact an eigenvalue problem (the eigenfunction is φ and the eigenvalue is c). Flierl (1979) shows that it is possible to find analytic approximations of φ and c under simplifying assumptions, introducing balances between linear and nondispersive effects, and between nonlinear and dispersive effects.

To achieve these balances, we impose:
Weak dispersion: x and y scales large compared to the Rossby radius of deformation.
Weak nonlinearity (due to steepening effect of the mean flow): this is achieved by making the vorticity and surface functionals nearly linear

$$P(Z, z) \sim A(z)\, Z + B(Z, z) \qquad \text{with } B << AZ$$

$$G(Z) \sim CZ + D(Z) \qquad \text{with } D << CZ$$

3. PURE TOPOGRAPHIC EFFECTS

The fact that curved topography can support solitary waves is not new (Clarke, 1971; Malanotte Rizzoli and Hendershott, 1980; Malanotte Rizzoli, 1980). The last reference clearly shows all basic features associated with planetary solitary waves over variable relief. We choose to discuss the case of pure topographic effects because of its simplicity. The set of eqs. 2.2 and 2.3 reduces to

$$0 = G_0(cy), \qquad \frac{N^2}{f_0} b = G_1(cy)$$

and thus

$$\nabla^2\varphi + \frac{\partial}{\partial z}\frac{f_0^2}{N^2}\frac{\partial}{\partial z}\varphi = \frac{\beta}{c}\varphi$$

$$\frac{\partial\varphi}{\partial z} = 0 \qquad \text{at } z = 0$$

$$\frac{\partial\varphi}{\partial z} = G_1(\varphi + cy) - G_1(cy) \qquad \text{at } z = -H$$

We suppose that the bottom variation is given by

$$b(y) = b'(0)\, y + \frac{b''}{2}(0)\, y^2$$

(the quadratic term must be retained to introduce the desirable nonlinearity in our problem). The boundary condition at the bottom may be written

$$\frac{\partial\varphi}{\partial z} = \frac{N^2}{f_0}\left\{b'(0)\,\frac{\varphi}{c} + b''(0)\,\frac{\varphi^2}{2c^2}\right\} \quad \text{at } z = -H$$

We now choose the scale large enough so that

$$|\nabla^2\varphi| << |\frac{\partial}{\partial z}\frac{f_0^2}{N^2}\frac{\partial\varphi}{\partial z}|$$

and introduce zero- and first-order approximations to the solitary wave streamfunction and the phase speed.
Lowest order balance (eigenvalue problem for c^0)

$$\frac{\partial}{\partial z}\frac{f_0^2}{N^2}\frac{\partial\psi^0}{\partial z} = \frac{\beta}{c^0}\varphi^0$$

$$\frac{\partial\varphi^0}{\partial z} = 0 \qquad \text{at } z = 0 \qquad (3.1)$$

$$\frac{\partial\varphi^0}{\partial z} = \frac{N^0}{f_0}\frac{b'(0)}{c^0}\varphi^0 \qquad \text{at } z = -H$$

The difference between our situation and previous analyses for a flat-bottom ocean is that the phase speed, to be determined, occurs both in the differential equation *and* in the boundary conditions. Some interesting effects are to be expected.
First order balance

$$\nabla^2\varphi^0 + \frac{\partial}{\partial z}\frac{f_0^2}{N^2}\frac{\partial}{\partial z}\varphi^1 = \frac{\beta}{c^0}\varphi^1 - \frac{\beta}{(c^0)^2}c^1\varphi^0 \qquad (3.2)$$

$$\frac{\partial \varphi^1}{\partial z} = 0 \qquad \text{at } z = 0$$

$$\frac{\partial \varphi^1}{\partial z} = \frac{N^2}{f_0}\left[\frac{b'(0)}{c^0}\psi^1 - b'(0)\frac{c^1}{(c^0)^2}\varphi^0 + \frac{b''(0)}{2(c^0)^2}(\varphi^0)^2\right] \qquad \text{at } z = -H$$

From (3.1), we deduce

$$\varphi^0(x, y, z) = \mathcal{Y}(x, y)\, F(z) \tag{3.3}$$

and we may impose

$$\frac{1}{H}\int_{-H}^{0} F^2(z)\, \mathrm{d}z = 1$$

We substitute (3.3) in (3.2), multiply by $(1/H)\, F(z)$ and integrate over depth; the use of (3.1) and (3.2) enables us to eliminate all the surface terms generated.

We finally obtain

$$\nabla^2\mathcal{Y} + \frac{c^1}{(c^0)^2}\left[\beta + \frac{f_0 b'(0)}{H}F^2(-H)\right]\mathcal{Y} - \frac{f_0 b''(0)}{H}\frac{F^3(-H)}{2(c^0)^2}\mathcal{Y}^2 = 0 \tag{3.4}$$

When the disturbance remains isolated we find an eigenvalue problem (eigenvalue is c^1 and eigenfunction is $\mathcal{Y}$); if a solution exists, the existence of solitary disturbances due to topographic variations is possible.

Equation 3.4 is the analog of eq. 3.4 in Flierl's article (horizontal shear case) and is a particular form of a wide class of equations, which we can write as

$$\nabla^2\mathcal{Y} + A\mathcal{Y} + B\mathcal{Y}^2 = 0 \tag{3.5}$$

(A and B are parameters; we shall see in Section 5 how they depend on flow and bottom characteristics).

We shall look for radially symmetric solutions which approach zero as $r \to \infty$ with all their derivatives in a monotonous way (this imposes $A < 0$). We therefore put

$$\mathcal{Y}(r) = K\frac{A}{B}G(x)$$

with

$$x = \sqrt{-A}\, r$$

which gives

$$\frac{\mathrm{d}^2 G}{\mathrm{d}x^2} + \frac{1}{x}\frac{\mathrm{d}G}{\mathrm{d}x} - G - KG^2 = 0 \tag{3.6}$$

$$G(0) = 1$$

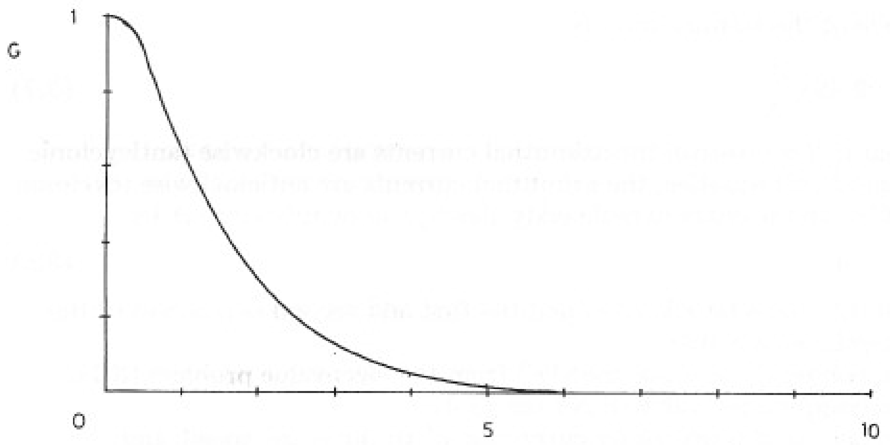

Fig. 1. Radial dependence of the dimensionless stream function.

$$\frac{dG}{dx}(0) = 0$$

The set of eqs. 3.6 is in fact an eigenvalue problem. The function $G(x)$ can be found numerically by standard methods. We find

$$K = -2.391$$

The radial dependence of the stream function $G(x)$ is shown by Figs. 1 and 2.

This result was first obtained by Flierl (1979). The presentation given above, however, seems less confused and more general to the author. The maximum

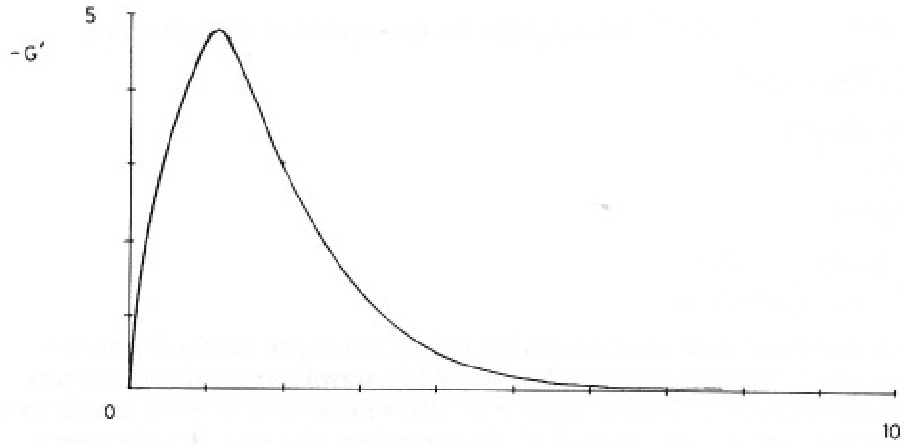

Fig. 2. Particle speed within the eddy (with reversed sign).

amplitude of the solitary wave is

$$\mathcal{Y}(0) = -2.391 \frac{A}{B} \tag{3.7}$$

Note that if B is positive, the azimuthal currents are clockwise (anticyclonic eddy) and if B is negative, the azimuthal currents are anticlockwise (cyclonic eddy). Clearly the characteristic eddy size (l) can be related to A by

$$l = 1/\sqrt{-A} \tag{3.8}$$

Given the characteristic size l and the first and second derivatives of the ocean depth, we are able:

(1) to compute the phase speed c^0 from the eigenvalue problem (3.1);
(2) to compute the value of B from (3.4);
(3) to deduce the first order correction c^1 to the phase speed; and
(4) to express the maximum amplitude $\mathcal{Y}(0)$.

From an immediate inspection of eqs. 3.4, 3.7 and 3.8, we may already deduce the following interesting features.

The shape and the extension of the solitary eddy are uniquely related to the second derivative of the depth profile, the amplitude $\mathcal{Y}(0)$ being inversely proportional to it; on the other hand, the sign of the second derivative conditions the cyclonic or anticyclonic character of the eddy.

The phase speed depends only on the first derivative (the bottom shape). We see that for $b'(0) > 0$ (shallow water to the north) the β effect is reinforced, which will augment the phase speed; for $b'(0)$, the situation is reversed. These results are in accordance with those obtained by Rhines (1970) for linear waves.

We now present two illustrating examples.

(1) $b'(0) = 0 \neq b''(0)$ and $N^2 = c^{te}$

It is straightforward to obtain

$$c^0 = -\beta R^2 \qquad \text{with } R = HN/\pi f_0 \text{ (the Rossby radius of deformation)}$$

$$F(z) = \sqrt{2} \cos \pi z/H$$

$$B = \frac{f_0 b''(0)\sqrt{2}}{H(c^0)^2}$$

$$c^1 = c^0 R^2/l^2$$

$$\mathcal{Y}(0) = \frac{2.391}{l^2} \frac{(c^0)^2 H}{f_0\sqrt{2}b''(0)}$$

The phase speed thus appears unaffected by the depth profile (compare with the result of Flierl, 1979, p. 23); the eddy travels westward. It appears that we will observe a cyclonic eddy if $b''(0)$ is negative (the eddy travels over a hill) and an anticyclonic eddy if $b''(0)$ is positive (the eddy travels over a depression).

If we take $H = 4000$ m, $N = 2 \times 10^{-3}$ s^{-1}, $f_0 = 5 \times 10^{-5}$ s^{-1}, $\beta = 2 \times 10^{-11}$ m^{-1} s^{-1}, $b''(0) = 10^{-10}$ m^{-1}, and $l = 200$ km, we obtain

$R = 51$ km,

$c^0 = -5$ cm s^{-1},

$\mathcal{Y}(0) = 91 \times 10^3$ m^2 s^{-1}

The last expression enables us to calculate the typical speed within the eddy

$$\mathcal{Y}(0) = U_0 l$$

and hence

$U_0 = 0.45$ m s^{-1}

which is a realistic speed for the description of certain observed eddies.

(2) $b'(0) \neq 0 \neq b''(0)$ and $N^2 = c^{te}$

We shall now concentrate our attention on the effects of the bottom variation on the phase speed. Two cases must be considered.

(a) $c^0 < 0$: westward propagation

The eigenvalue problem (3.1) is easily shown to reduce to

$$c^0 = -\frac{N^2H^2}{f_0^2 m^2}\beta \tag{3.9}$$

where

$$\frac{tgm}{m} = -\frac{b'(0)\, f_0}{\beta H}$$

A remark is necessary here: our approximation is only valid if

$$\left|\frac{\partial}{\partial z}\frac{f_0^2}{N^2}\frac{\partial}{\partial z}\varphi\right| >> |\nabla^2 \varphi|$$

This may be rewritten

$$\frac{f_0^2}{H^2N^2} m^2 >> \frac{1}{l^2}$$

or (using the Rossby radius of deformation)

$$m^2 >> \pi^2 \frac{R^2}{l^2}$$

This last expression indicates that we must be careful in our analysis: values of m that are too small must be rejected (the phase speed c^0 would increase — in modulus — without limit, which is unacceptable).

Therefore, we shall only retain values of m, solutions of (3.9), but belong-

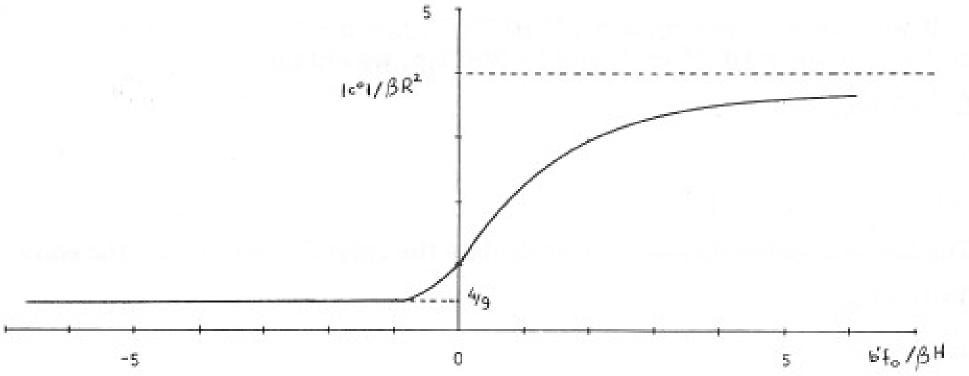

Fig. 3. The zero-order dimensionless phase speed as a function of the dimensionless slope.

ing to [$+\pi/2$, $3\pi/2$]. Moreover, this ensures a continuous dependence of c^0 in $b'(0)$.

If we choose $b'(0) = 10^{-3}$, keeping the same values as before for the other parameters, we have (this shape corresponds to 100 m/100 km) $m = 2.2$, and $c^0 = -11$ cm s^{-1} (9.5 km day^{-1}).

Figures 3 and 4 show the dependence of c^0 and c^1 on the bottom slope $b'(0)$. We observe that the phase speed is increased for a bottom sloping to the north (the β effect is reinforced) while it is decreased for a bottom sloping to the south; for $b'(0) = 0$, we recover the results of the previous example.

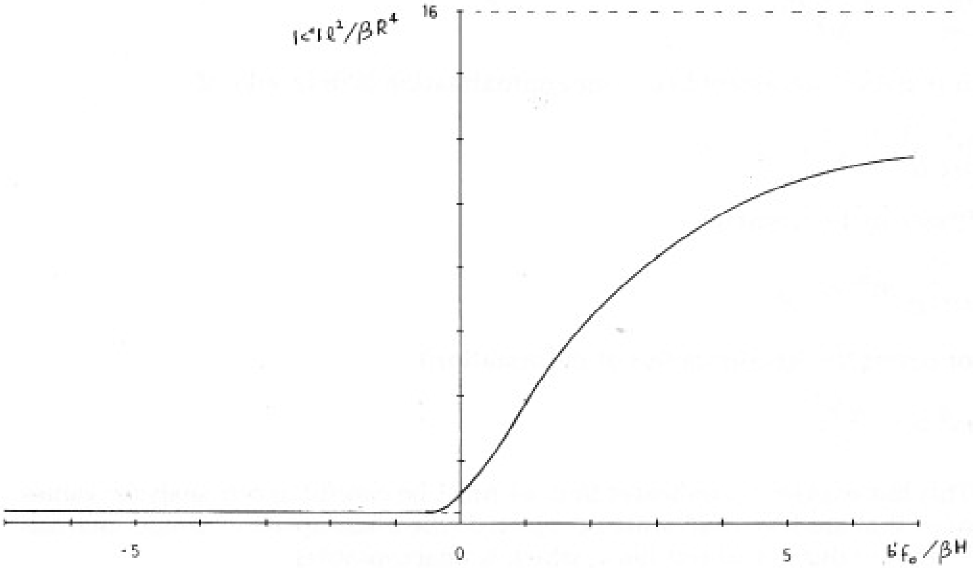

Fig. 4. The first-order dimensionless phase speed as a function of the dimensionless slope.

Note the strong influence of a positive slope: the speed can be multiplied by a factor $4(c^0)$ or $16(c^1)$; the influence of a negative slope is much less important.

(b) $c^0 > 0$: eastward propagation

The presence of a bottom slope can force solitary eddies to travel eastward; mathematically, the origin of this lies in (3.1) where the eigenvalue c^0 appears both in the equation and in the boundary conditions; from a physical point of view, these solitary waves represent nonlinear extensions of Rhines' linear results (Rhines, 1970).

Equations 3.1 give

$$\frac{thm}{m} = -\frac{b'(0)\,f_0}{\beta H} \tag{3.10}$$

$$c^0 = \frac{N^2H^2}{f_0^2m^2}\,\beta$$

For a solution to exist, we must have

$$b'(0) < 0 \qquad \text{and} \qquad |b'(0)|\frac{f_0}{\beta H} < 1$$

i.e., the northern region must be that of deepest water and the slope must not be too large.

With the parameters given above, this yields

$$|b'(0)| < 1.6 \times 10^{-3} \ (= 160 \text{ m}/100 \text{ km})$$

If the two conditions given above are not satisfied, no eastward propagation will take place. In the present case, the vertical eigenfunction $F(z)$ will be a hyperbolic cosine, with argument mz/H.

4. INTERACTION WITH AN ARBITRARY FLOW

Our analysis being strictly parallel to that of Flierl (1979), we shall partition the mean flow streamfunction Ψ_* (* designates dimensional quantities) into a part representing north—south invariant translation and a nonuniform part. We also work with nondimensional equations, so we write

$$\Psi_* = -\beta R^2 L U(z)\, Y + \Delta U_0 L^2/l\hat{\Psi}(Y, z)$$

where

L is the mean flow scale

l is the eddy scale

ΔU_0 is the variation of the mean flow velocity over the eddy scale

$Y = y_*/l \qquad z = z_*/H$

R is the Rossby radius of deformation

It is also convenient to take

$c_* = \beta R^2 c$

$P_*(Z_*, Z_*) = \beta L P(Z_*/\beta R^2 L, z_*/H)$

$G_*(Z_*) = \dfrac{\beta R^2 L}{H} G(Z_*/\beta R^2 L)$

$b_*(y_*) = Hb(Y)$

The eddy stream function φ needs a different scaling

$\varphi_* = U_0 l \varphi(x, y, z) \qquad (x, y) = {}^1/_l\,(x^*, y^*)$

and several nondimensional parameters can be introduced

$\hat{\epsilon} = \Delta U_0/\beta R^2$

$\epsilon = U_0/\beta R^2$

$\delta = l/L$

$\gamma = R^2/l^2$

and

$S = N^2 H^2/f_0^2 R^2 \qquad$ (Burger's number)

With this choice, eqs. 2.2 and 2.3 become:
(1) For the mean flow (definition of functional P)Ψ (Y, z)

$$P\left[Y(c-U) + \frac{\hat{\epsilon}}{\delta}\hat{\Psi}, z\right] = Y\left(1 - \frac{\partial}{\partial z}\frac{1}{S}\frac{\partial U}{\partial z}\right) + \frac{\hat{\epsilon}}{\delta}\left(\frac{\partial}{\partial z}\frac{1}{S}\frac{\partial}{\partial z}\hat{\Psi} + \gamma\delta^2\frac{\partial^2\hat{\Psi}}{\partial y^2}\right) \qquad (4.1)$$

(2) For the perturbation $\varphi(x, y, z)$

$$\gamma\nabla^2\varphi + \frac{\partial}{\partial z}\frac{1}{S}\frac{\partial}{\partial z}\varphi = \frac{1}{\epsilon\delta}\left\{P\left[\epsilon\delta\varphi + (c-U)\,\delta y + \frac{\hat{\epsilon}}{\delta}\hat{\Psi}, z\right]\right.$$
$$\left. - P\left[\delta y(c-U) + \frac{\hat{\epsilon}}{\delta}\hat{\Psi}, z\right]\right\} \qquad (4.2)$$

The boundary conditions are then:
(1) For the mean flow (definitions for G_0 and G_1)

$$-Y\frac{dU}{dz} + \frac{\hat{\epsilon}}{\delta}\frac{\partial\hat{\Psi}}{\partial z} = G_0\left[Y(c-U) + \frac{\hat{\epsilon}}{\delta}\hat{\Psi}\right] \qquad \text{at } z = 0$$
$$= -\frac{Sf_0}{L}b(y) + G_1\left[Y(c-U) + \frac{\hat{\epsilon}}{\delta}\hat{\Psi}\right] \qquad \text{at } z = -1 \qquad (4.3)$$

(2) For the perturbation

$$\frac{\partial\varphi}{\partial z}=\frac{1}{\epsilon\delta}\left\{G_0\left[\epsilon\delta\varphi+(c-U)\,\delta y+\frac{\hat{\epsilon}}{\delta}\hat{\Psi}\right]-G_0\left[\delta y(c-U)+\frac{\hat{\epsilon}}{\delta}\hat{\Psi}\right]\right\} \tag{4.4}$$

at $z = 0$

and a similar expression for G_1 at $z = -1$

$\varphi \to 0$ as $|x| \to +\infty$

In order to maintain a weak dispersion, strong eddies and small changes of the mean flow across the eddy, we must impose

$\gamma << 1$, $\epsilon \sim 1/\gamma$, and $\hat{\epsilon} \sim \gamma^2$.

In Flierl's work, two cases are considered: very large-scale mean flow ($\delta \sim \gamma^2$) and intermediate-scale mean flow ($\delta \sim \gamma$). We shall follow his analysis in the first case only and explain the effects of the topographic variations.

5. INFLUENCE OF A VERY LARGE MEAN FLOW AND BOTTOM EFFECTS ($\delta \sim \gamma^2$)

We require that

$\dfrac{dU}{dz} = 0 = \dfrac{\partial\hat{\Psi}}{\partial z}$ at top and bottom

$$\hat{\Psi}(0, z) = 0 = \frac{\partial\hat{\Psi}}{\partial y}(0, z)$$

These assumptions do not appear to be so restrictive and simplify further developments considerably.

This implies

$$G_0 = 0 = \frac{\partial\varphi}{\partial z} \qquad \text{at } z = 0$$

and

$$G_1\left[Y(c-U)+\frac{\hat{\epsilon}}{\delta}\hat{\Psi}\right]=\frac{Sf_0}{\beta L}b(Y) \qquad \text{at } z=-1 \tag{5.1}$$

It is then possible to develop the boundary constraints in powers of γ: expression (4.4) becomes (correct to order γ)

$$\frac{\partial\varphi}{\partial z}=G_1'(0)\,\varphi+\tfrac{1}{2}G_1''(0)\,\frac{\epsilon\delta}{\gamma}\,\gamma\varphi^2 \qquad \text{at } z=-1 \tag{5.2}$$

with

$$G_1'(0) \equiv \frac{dG_1}{dZ}(Z) \qquad \text{for } Z = 0$$

$$G_1''(0) \equiv \frac{d^2 G_1}{dZ^2}(Z) \qquad \text{for } Z = 0$$

We can evaluate G_1' and G_1'' by taking Y derivatives of (4.3); we obtain

$$G_1'(0) = \frac{Sf_0}{\beta L}\frac{db}{dY}(0)\frac{1}{c - U(-1)}$$

$$G_1''(0) = \frac{1}{[c - U(-1)]^2}\frac{Sf_0}{\beta L}\left[\frac{d^2 b}{dY^2}(0) - \frac{db}{dY}(0)\frac{\hat{\epsilon}}{\delta}\frac{\hat{\Psi}''(0,-1)}{c - U(-1)}\right] \tag{5.3}$$

$$\left(\hat{\Psi}'' \equiv \frac{\partial^2 \hat{\Psi}}{\partial Y^2}\right)$$

Expression (4.2), describing the evolution of φ, can also be expressed in powers of γ; this expansion is not affected by the boundary conditions and must be identical to that deduced by Flierl, so

$$\gamma\nabla^2\varphi + \frac{\partial}{\partial z}\frac{1}{S}\frac{\partial\varphi}{\partial z} = \frac{1 - [(d/dz)(1/S)(dU/dz)]}{c - U}\varphi$$
$$+ \gamma\left(\frac{\epsilon\hat{\epsilon}}{\gamma}\right)\frac{1}{2}\varphi^2\left\{-\frac{1 - [(d/dz)(1/S)(dU/dz)]}{(c-U)^3}\hat{\Psi}''(0,z)\right.$$
$$\left. + \frac{(\partial/\partial z)(1/S)(\partial/\partial z)\,\hat{\Psi}''(0,z)}{(c-U)^2}\right\} \tag{5.4}$$

The system of PDE obtained in (S.1), (S.2), (S.3) and (S.4) is now closed; we solve it by introducing expansions in powers of γ for φ and c

$$\varphi = \varphi^0 + \gamma\varphi^1 + \ldots$$
$$c = c^0 + \gamma c^1 + \ldots$$

and we have:
Lowest order balance: linear and dispersive

$$\frac{\partial}{\partial z}\frac{1}{S}\frac{\partial}{\partial z}\varphi^0 = \frac{1 - [(d/dz)(1/S)(dU/dz)]}{c^0 - U}\varphi^0$$

$$\frac{\partial\varphi^0}{\partial z} = 0 \qquad \text{at } z = 0 \tag{5.5}$$

$$\frac{\partial\varphi^0}{\partial z} = \frac{Sf_0}{\beta L}\frac{db(0)}{dY}\frac{1}{c^0 - U(-1)}\varphi^0 \qquad \text{at } z = -1$$

This implies that

$$\varphi^0 = \mathcal{Y}(x, y)\,\mathcal{F}(z)$$

with

$$\frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{S}\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}z} = \frac{1-[(\mathrm{d}/\mathrm{d}z)(1/S)(\mathrm{d}U/\mathrm{d}z)]}{c^0 - U}\mathcal{F}$$

$$\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}z} = 0 \qquad \text{at } z = 0$$

$$\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}z} = \frac{f_0 S}{\beta L}\frac{\mathrm{d}b(0)}{\mathrm{d}Y}\frac{1}{c^0 - U(-1)}\mathcal{F} \qquad \text{at } z = -1$$

We are then free to impose

$$\int_{-1}^{0} \mathcal{F}^2(z)\,\mathrm{d}z = 1$$

Order 1 in γ: nonlinear balance

$$\mathcal{F}\cdot\nabla^2\mathcal{Y} + \frac{\partial}{\partial z}\frac{1}{S}\frac{\partial}{\partial z}\varphi^1 = \frac{1-[(\mathrm{d}/\mathrm{d}z)(1/S)(\mathrm{d}U/\mathrm{d}z)]}{c^0 - U}\varphi^1$$

$$-\frac{c^1}{(c^0-U)^2}\left(1-\frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{S}\frac{\mathrm{d}U}{\mathrm{d}z}\right)\mathcal{Y}\mathcal{F} + \frac{\epsilon\hat{\epsilon}}{\gamma}\frac{1}{2}\mathcal{Y}^2\mathcal{F}^2\left[\frac{1}{(c^0-U)^2}\frac{\partial}{\partial z}\frac{1}{S}\frac{\partial}{\partial z}\hat{\Psi}''(0,z)\right.$$

$$\left. -\frac{1}{(c^0-U)^3}\left(1-\frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{S}\frac{\mathrm{d}U}{\mathrm{d}z}\right)\hat{\Psi}''(0,z)\right] \tag{5.6}$$

and

$$\frac{\partial\varphi^1}{\partial z} = 0 \qquad \text{at } z = 0$$

$$\frac{\partial\varphi^1}{\partial z} = \frac{f_0 S}{\beta L}\frac{\mathrm{d}b(0)}{\mathrm{d}Y}\frac{1}{c^0-U(-1)}\left(\varphi^1 - \frac{c^1}{c^0-U(-1)}\varphi^0\right) \tag{5.7}$$

$$+\frac{1}{2}\frac{f_0 S}{\beta L}\frac{1}{[c^0-U(-1)]^2}\left[\frac{\mathrm{d}^2 b(0)}{\mathrm{d}Y^2} - \frac{\hat{\epsilon}}{\delta}\frac{\mathrm{d}b(0)}{\mathrm{d}Y}\hat{\Psi}''(0,-1)\frac{1}{c_0-U(-1)}\right]\frac{\epsilon\delta}{\gamma}(\varphi^0)^2$$

The solvability condition follows from multiplying (5.6) by $\mathcal{F}$ and integrating over depth; using the boundary conditions (5.7) we observe that the φ^1 terms *do not* drop out, but generate additional terms, proportional to G and G^2.

Our final equation, to be compared to (4.12) in Flierl, is

$$\nabla^2\mathcal{Y} + c^1\mathcal{Y}\left\{\frac{f_0}{\beta L}\frac{1}{[c^0-U(-1)]^2}\frac{\mathrm{d}b(0)}{\mathrm{d}y}\mathcal{F}^2(-1) + \int_{-1}^{0}\mathcal{F}^2\frac{1-[(\mathrm{d}/\mathrm{d}z)(1/S)(\mathrm{d}U/\mathrm{d}z)]}{(c^0-U)^2}\,\mathrm{d}z\right\}$$

$$+\frac{1}{2}\mathcal{Y}^2\left[\frac{\epsilon\hat{\epsilon}}{\gamma}\int_{-1}^{0}\mathcal{F}^3\left\{\frac{1-[(\mathrm{d}/\mathrm{d}z)(1/S)(\mathrm{d}U/\mathrm{d}z)]}{(c^0-U)^3}\hat{\Psi}''(0,z) - \frac{\frac{\partial}{\partial z}\frac{1}{S}\frac{\partial}{\partial z}\hat{\Psi}''(0,z)}{(c^0-U)^2}\right\}\mathrm{d}z\right.$$

$$\left.-\frac{f_0}{\beta L}\frac{\epsilon\delta}{\gamma}\frac{1}{[c^0-U(-1)]^2}\left[\frac{\mathrm{d}^2 b(0)}{\mathrm{d}Y^2} - \frac{\hat{\epsilon}}{\delta}\frac{\mathrm{d}b(0)}{\mathrm{d}Y}\frac{\hat{\Psi}''(0,-1)}{c^0-U(-1)}\right]\mathcal{F}^3(-1)\right] = 0 \tag{5.8}$$

and

$\mathcal{Y} \to 0 \qquad \text{for } |x| \to +\infty$

Equation 5.8) shows the combined effects of topographic variations and mean flow interaction. It appears to be quite complex in form, but we immediately observe that it belongs to the class of equations defined by expression (3.5); note, however, that (5.8) is already nondimensional.

On the other hand, the phase speed c^0 appears as eigenvalue of (5.5), the exact analogue of expression (3.1), except for the presence of U. The remarks that have been made in Section 3 remain valid here, in particular those concerning the importance of the bottom slope.

If we wish to write down the radially symmetric solutions — the more interesting ones — we can follow the same procedure as in Section 3 (provided we stay within the limits of our assumptions):

(1) Given L (characteristic length of the mean flow), l (characteristic length of the eddy) and ΔU_0 (the variation of the mean flow on the eddy scale l), we deduce $\hat{\epsilon}$, δ and γ.

(2) From (5.5) we obtain the adimensional phase speed c^0; note that it depends only on the bottom slope $b'(0)$ and on the north—south invariant translation $U(z)$.

We saw in Section 3 that a sloping bottom has a non-negligible effect on the phase speed (recall the second example given in Section 3) and that new phenomena can take place (eastward propagation and boundary trapped buoyancy waves).

(3) From (5.7), it follows that

$$c^1 = -\left\{\frac{f_0}{\beta L}\frac{1}{[c^0-(-1)]^2}\frac{\mathrm{d}b}{\mathrm{d}Y}(0)\,\mathcal{F}^2(-1) + \int_{-1}^{0}\mathcal{F}^2\,\frac{1-[(\mathrm{d}/\mathrm{d}z)(1/S)(\mathrm{d}U/\mathrm{d}z)]}{(c^0-U)^2}\,\mathrm{d}z\right\}^{-1}$$

The same remarks hold for c^1.

(4) Finally we obtain information on the particle speed within the eddy and on on the sign of it (cyclonic or anticyclonic); we must impose

$$\frac{1}{2}\left[\frac{\epsilon\hat{\epsilon}}{\gamma}\int_{-1}^{0}\mathcal{F}^3\left\{\frac{1-[(\mathrm{d}/\mathrm{d}z)(1/S)(\mathrm{d}U/\mathrm{d}z)]}{(c^0-U)^3}\hat{\Psi}''(0,z)-\frac{\frac{\partial}{\partial z}\frac{1}{S}\frac{\partial}{\partial z}\hat{\Psi}''(0,z)}{(c^0-U)^2}\right\}\mathrm{d}z\right.$$

$$\left. -\frac{f_0}{\beta L}\frac{\epsilon\delta}{\gamma}\frac{1}{[c^0-U(-1)]^2}\left[\frac{\mathrm{d}^2b}{\mathrm{d}Y^2}(0)-\frac{\epsilon}{\delta}\frac{\mathrm{d}b}{\mathrm{d}Y}(0)\,\frac{\hat{\Psi}''(0,-1)}{c^0-U(-1)}\right]\mathcal{F}^3(-1)\right] = 2.391 \qquad (5.9)$$

This last expression gives the nondimensional amplitude ϵ. A negative value for the amplitude corresponds to a cyclonic eddy while a positive value corresponds to an anticyclonic eddy; from ϵ, the values of the particle speed within the eddy can be obtained.

Expression (5.9) shows that ϵ depends in a complex way on the bottom slope and on U, the north–south invariant part of the mean flow speed, but also on the second derivative of the depth and on the nonuniform part of the mean flow streamfunctions — two parameters which played no part in the phase speed determination. We may partition the dependence of ϵ into three parts:

(1) a first part representing pure mean flow effects [these are fully analyzed in the work of Flierl (1979)];

(2) a second part describing the effect of depth variation only (in terms of b'') (these effects are examined in Section 3); and

(3) a third part due to the interaction between the mean flow and the bottom slope.

The presence of so great a number of parameters and functions in eqs. 5.8 and 5.9 makes them so complex that the perspective of a complete study appears to be impracticable. The usefulness of writing them, however, appears in the fact that they permit us to compare the relative influence of the bottom and of the mean flow. Two extreme cases were presented: no bottom effect (see Flierl, 1979) and no mean flow effect (see Section 3 of this work). It seems that both effects must often be taken into account to give a more physically acceptable solution to the problem of nonlinear eddies. The example that follows is illustrative of the possible cooperative or non-cooperative effects of a bottom slope and a mean shear flow. If we take

$\hat{\Psi} = \hat{\Psi}(Y)$ only (shear flow, barotropic)

$U = 0$

$b' \neq 0 = b''$ (bottom slope)

and

$N^2 \neq c^{te}$

we shall encounter a nonlinear effect due to the shear flow, and a nonlinear effect due to the bottom slope/mean flow interaction.

The coefficient multiplying the nonlinear term in (5.7) is quickly evaluated (the reader is referred to Section 3 for the phase speed calculation)

$$\frac{1}{2}\frac{1}{(c^0)^3}\frac{\epsilon\hat{\epsilon}}{\gamma}\hat{\Psi}''(0)\left[\int_{-1}^{0}\mathcal{F}^3(z)\,\mathrm{d}z + \frac{f_0}{\beta L}\frac{\mathrm{d}b}{\mathrm{d}Y}(0)\,\mathcal{F}^3(-1)\right]$$

The existence of a solitary eddy is possible if this expression equals 2.391; if the term between the brackets is very small (i.e., if the nonlinearity due to external features such as the mean flow and the bottom slope is small) then ϵ, the nonlinearity due to the eddy amplitude, must be very large. This will not always be possible (the particle speed must take acceptable values) and the problem is in fact linear.

This will be the case when

$$\xi = \int_{-1}^{0} \mathcal{F}^3 \, \mathrm{d}z \simeq -\frac{f_0}{\beta L} \frac{\mathrm{d}b}{\mathrm{d}Y}(0)\, \mathcal{F}^3(-1) \tag{5.9}$$

Equation 5.9 defines a critical value for the bottom slope: in the vicinity of this value we expect linear wave dynamics. For a slope below this critical value, the effects of the bottom slope and of the shear flow are cooperative if we assume (mid-ocean situations) $\xi > 0$ and $\mathcal{F}^3(-1) < 0$; if the slope is beyond this value, the effects are opposite.

6. SUMMARY

The description of the amplitude and speed of a baroclinic solitary wave in the presence of bottom variations and interacting with a large-scale mean flow involves an eigenvalue problem of the type

$$\nabla^2 G + c^1 A G + B G^2 = 0$$

$$G \to 0 \qquad \text{as } |x| \to +\infty$$

where $G(x, y)$ is the horizontal shape; c^1 is the order R^2/l^2 correction to phase speed, R being the Rossby deformation radius and l the characteristic eddy scale; and A and B are functions of the perturbed large scale mean flow and of $\mathcal{F}$, the vertical structure of the eddy. A is also a function of the prime derivative of b, the bottom deviation, with respect to the north—south direction; B depends on the prime and second derivatives of b. Since A is correlated to c^1 and B to the maximum amplitude of the wave, it is possible to deduce the effects of bottom variations on these characteristics.

The special role played by the bottom slope in the boundary conditions allows a new type of solitary wave to exist (with eastward propagation); this peculiarity had already been shown for linear waves by Rhines (1970).

Finally, it appears that the combined effects can be either cooperative or antagonistic: the nonlinearity — necessary to the existence of isolated disturbances — is affected by both effects. The topographic situation must be taken into account to decide if solitary eddies may exist or if linear dynamics are relevant.

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