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Periodic solutions and recurrence for nonlinear Schrödinger equation: a Fourier-mode approach

by

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RÉSUMÉ. — On étudie la stabilité de solutions uniformes de l'équation non linéaire de Schrödinger par un modèle faisant intervenir un nombre limité de composantes de Fourier.
Un système d'équations différentielles couplées peut être obtenu, décrivant l'effet des modulations. L'instabilité de Benjamin et Feir apparait comme univoquement liée à la technique de linéarisation. Une solution exacte est trouvée, qui confirme le phénomène de récurrence Fermi-Pasta-Ulam, gouvernant l'évolution à long terme.

ABSTRACT. — The stability of uniform solutions of the nonlinear Schrödinger equation is tested by means of a three-mode model, involving the interaction of a finite number of Fourier components.
A set of coupled differential equations is obtained, describing the effect of sideband modulations. The Benjamin-Feir instability appears clearly related to the linearization technique. An exact solution is found, which confirms the Fermi-Pasta-Ulam recurrence, governing the long-time evolution.

1. Introduction

We consider the non-dimensional form of the nonlinear Schrödinger equation (NLS):

$$i\frac{\partial a}{\partial \theta}-\frac{1}{8}\frac{\partial^2 a}{\partial y^2}-\frac{1}{2}|a|^2 a=0. \tag{1}$$

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This equation has been widely used to describe the evolution of nonlinear and weakly dispersive envelopes (e. g. Whitham [1], [2]; Lake and Yuen [3]).

Using the inverse scattering method, Zakharov and Shabat [4] solved the initial value problem. However, for initial conditions which do not decay sufficiently rapidly as $|y| \to \infty$, this method cannot be applied.

We shall consider here uniform and periodic solutions of the type:

$$a = a_0 \exp\left(-\frac{i}{2} a_0^2 \theta\right), \tag{2}$$

which exhibit two known properties:

1) the Benjamin and Feir instability (Benjamin and Feir [5]): the solution (2) is unstable to infinitesimal perturbations of the form:

$$\{a_+ \exp(i\omega\theta + i\varkappa y) + a_- \exp(-i\omega\theta - i\varkappa y)\} \exp\left(-\frac{i}{2} a_0^2 \theta\right), \tag{3}$$

provided that:

$$0 < \varkappa < 2\sqrt{2}\, a_0.$$

This property has been shown by Benjamin and Feir (1967), when investigating the behaviour of deep water waves (Stokes' waves) with sideband interaction;

2) the Fermi-Pasta-Ulam recurrence: the unstable modulations, as discovered by Benjamin and Feir, are the first stage of instability; in the next stage, the solution tends to demodulate and return to the initial state.

2. General purpose

Our purpose here is to examine equation (1) by means of an approximate theory involving the interaction of a finite number of Fourier modes.

For a finite number of Fourier components, we can write:

$$a(y, \theta) = \frac{1}{2} \sum_{\nu} \varphi_\nu(\theta) \exp(ik_\nu y), \tag{4}$$

where ν runs over $0, \pm 1, \ldots, \pm N$ (N fixed integer).

When we substitute (4) into (1), we have, for each of the original components:

$$i\frac{d\varphi_\nu}{d\theta}+\frac{1}{8}k_\nu^2\,\varphi_\nu=\frac{1}{8}\sum_{k_\alpha-k_\beta+k_\gamma=k_\nu}\varphi_\alpha\,\varphi_\beta^*\,\varphi_\gamma. \tag{5}$$

Note that the cubic term also generates new components to be added to (4). They will be neglected in the first approximation (they do not resonate, at least at the third order).

2.1. The Stokes' wave as a self-interaction product

We consider the only mode $k_\nu=0$, which gives:

$$a=\frac{1}{2}\varphi_0(\theta),$$

$$i\frac{d\varphi_0}{d\theta}=\frac{1}{8}|\varphi_0|^2\,\varphi_0,$$

and:

$$a=a_0\exp\left(-\frac{i}{2}a_0^2\,\theta\right).$$

This is the exact solution of NLS equation, generating the Stokes' wave.

2.2. The effect of sideband modulations

We are now able, by means of our analysis:

1) to recover the results of Benjamin and Feir;

2) to predict the occurrence of the FPU phenomenon, and to explain its relation with the B – F instability.

Let us examine the case of a perturbed solution of the form:

$$a=\frac{1}{2}\{\varphi_0(\theta)+\varphi_+(\theta)\exp i\mu y+\varphi_-(\theta)\exp(-i\mu y)\}. \tag{6}$$

This solution consists of a central mode with side-band interaction.

Substituting in (5), we have:

$$(7)\quad \begin{cases} i\dfrac{d\varphi_0}{d\theta}=\dfrac{1}{8}\{\varphi_0(|\varphi_0|^2+2|\varphi_+|^2+2|\varphi_-|^2)+2\varphi_0^*\varphi_+\varphi_-\}, \\ i\dfrac{d\varphi_+}{d\theta}+\dfrac{1}{8}\mu^2\varphi_+=\dfrac{1}{8}\{\varphi_+(2|\varphi_0|^2+|\varphi_+|^2+2|\varphi_-|^2)+\varphi_0^2\varphi_-^*\}, \end{cases}$$

and a similar equation for φ_-.

To take out the main oscillation, we introduce:

$$\varphi_0=2A_0,$$

$$\varphi_\pm=2A_\pm\exp\left(\frac{i\mu^2\theta}{8}\right).$$

We then linearize the last set of equations to obtain (on the basis that $|A_\pm|\ll|A_0|$):

$$(8)\quad \begin{cases} i\dfrac{dA_0}{d\theta}=\dfrac{1}{2}|A_0|^2A_0, \\ i\dfrac{dA_+}{d\theta}=\dfrac{1}{2}A_0^2\exp\left(-\dfrac{i\mu^2\theta}{4}\right)A_-^*+|A_0|^2A_+, \end{cases}$$

and a similar equation for A_-.

We then have solutions in the form:

$$A_0=a_0\exp\left(-\frac{i}{2}a_0^2\theta\right),$$

$$A_\pm=a_\pm\exp\ (+i\lambda_\pm\theta),$$

with:

$$\lambda_\pm^2+\left(a_0^2+\frac{\mu^2}{4}\right)\lambda_\pm+\frac{a_0^2}{2}\left(\frac{a_0^2}{2}+\frac{\mu^2}{2}\right)=0.$$

We deduce that $\lambda_\pm$ takes a complex value (and instability emerges) if:

$$a_0^2-\frac{\mu^2}{8}>0,$$

or:

$$0<\mu<2\sqrt{2}\,a_0,$$

which is the Benjamain and Feir result.

It is possible to pursue the analysis of equations (7) further *without* linearizing; three first integrals of these equations may be found; two integrals are rapidly obtained:

$$2|\varphi_\pm|^2+|\varphi_0|^2=C_\pm \text{ (real positive Const.)}, \tag{9}$$

but the third one is more complicated; it can be deduced as follows: the NLS equation can be derived from the Lagrangian density:

$$L=\frac{i}{2}(a^* a_\theta - a a_\theta^*)+\frac{1}{8}|a_y|^2-\frac{1}{4}|a|^4. \tag{10}$$

We substitute (6) in this expression and a variational principle for equations (7) is found, with Lagrangian:

$$L=i\left(\varphi_0^*\frac{d\varphi_0}{d\theta}+\varphi_+^*\frac{d\varphi_+}{d\theta}+\varphi_-^*\frac{d\varphi_-}{d\theta}-\text{complex conjugate}\right)$$
$$+\frac{\mu^2}{4}(|\varphi_+|^2+|\varphi_-|^2)-\frac{1}{8}(|\varphi_0|^4+|\varphi_+|^4+|\varphi_-|^4+4|\varphi_0|^2|\varphi_+|^2$$
$$+4|\varphi_0|^2|\varphi_-|^2+4|\varphi_+|^2|\varphi_-|^2+2F),$$

where:

$$F=\varphi_0\varphi_0\varphi_+^*\varphi_-^*+\varphi_0^*\varphi_0^*\varphi_+\varphi_-.$$

This Lagrangian is invariant to translations in time, so we have:

$$H=\mu^2(|\varphi_+|^2+|\varphi_-|^2)-\left(\frac{1}{2}\right)(|\varphi_0|^4+|\varphi_+|^4+|\varphi_-|^4+4|\varphi_0|^2|\varphi_+|^2$$
$$+4|\varphi_0|^2|\varphi_-|^2+4|\varphi_+|^2|\varphi_-|^2+2F)=\text{real Const.}$$

We can now write a single equation for $|\varphi_0|^2$ because:

(i) F is a function of $|\varphi_0|^2$, $|\varphi_+|^2$, $|\varphi_-|^2$ alone;

(ii) $|\varphi_+|^2$ and $|\varphi_-|^2$ are functions of $|\varphi_0|^2$ alone [from (9)];

(iii) we have:

$$\frac{id|\varphi_0|^2}{d\theta}=\frac{1}{4}(\varphi_0^*\varphi_0^*\varphi_+\varphi_- - \varphi_0\varphi_0\varphi_+^*\varphi_-^*),$$

and if we square this expression, we have:

$$\left(\frac{d|\varphi_0|^2}{d\theta}\right)^2=\frac{1}{4}|\varphi_0|^4|\varphi_+|^2|\varphi_-|^2-\frac{F^2}{16}=P(|\varphi_0|^2), \tag{11}$$

the right-hand side being a quartic in $|\varphi_0|^2$, solutions may be found in elliptic functions.

Introducing:

$$x=|\varphi_0|^2,$$

$$x_0=x \quad \text{when} \quad \theta=0,$$

$$Z=2\arg\varphi_0-\arg\varphi_+-\arg\varphi_- \quad \text{when} \quad \theta=0,$$

we have:

$$\left(\frac{dx}{d\theta}\right)^2=P(x;\,x_0,\,C_+,\,C_-,\,\mu,\,Z), \tag{12}$$

with:

$$P=\frac{1}{16}\{x^2(C^+-x)(C_--x)-F^2\},$$

and:

$$F=(x_0-x)\left\{\mu^2+\frac{C_++C_-}{4}-\frac{3}{4}(x+x_0)\right\}+x_0\sqrt{(C_+-x_0)(C_--x_0)}\cos Z.$$

The domain of physical interest is $0\leqq x\leqq \min(C_+,\,C_-)$.

We note also that:

- $P(x_0)\geqq 0$;
- $P(x_0)=0$, if and only if $Z=n\pi$;
- $P(0)\leqq 0$, and $P(C_\pm)\leqq 0$;
- $\lim\limits_{\pm\infty} P(x)=+\infty$.

The special form of equation (12) allows us to discuss all possible behaviours for x (i. e. $|\varphi_0|^2$): in rational mechanics, equation (12) is in fact the expression of conservation of energy for a particle under the influence of the quartic potential $-P(x)$ with suitable initial conditions.

Keeping in mind the special properties of P, we discover that only three different behaviours are possible (depending on the various parameters):

1) x_0 lies between two simple roots of P: $a\leqq x_0\leqq b$ (*fig.* 1). The evolution of $|\varphi_0|^2$ will be periodic in time; the evolution of the perturbations will be

periodic too, with the same period. In other words, the energy oscillates between the modes. This is F.P.U. recurrence. Note that b is always lower than min (C_+, C_-);

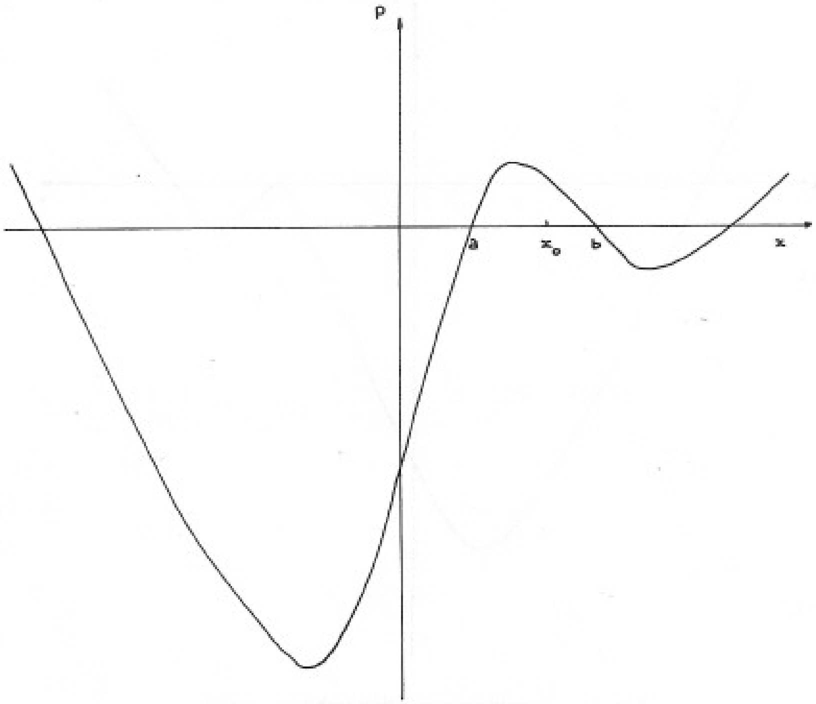

Fig. 1. – Periodic behaviour (arbitrary scale).

2) x_0 is a double root of P (*fig.* 2); in this case, $|\varphi_0|^2$ remains constant in time, together with the perturbations. (This is the analog of a "stable equilibrium" in rational mechanics.) Only the phases are altered;

3) x_0 lies between a simple root on its left, and a double root on its right; this double root is nececcarily $C_+ = C_-$ (*fig.* 3). Here $|\varphi_0|^2$ will tend in an asymptotic way ($t \to +\infty$) to the value $C_+ = C_-$; the perturbations die away: energy comes back to the central mode.

This will be the case if:

$$C_+ = C_- = 4\left[\mu^2 + \frac{x_0}{4} - 2\,x_0 \cos^2\left(\frac{Z}{2}\right)\right].$$

This case is a very special one, in which the period tends to infinity.

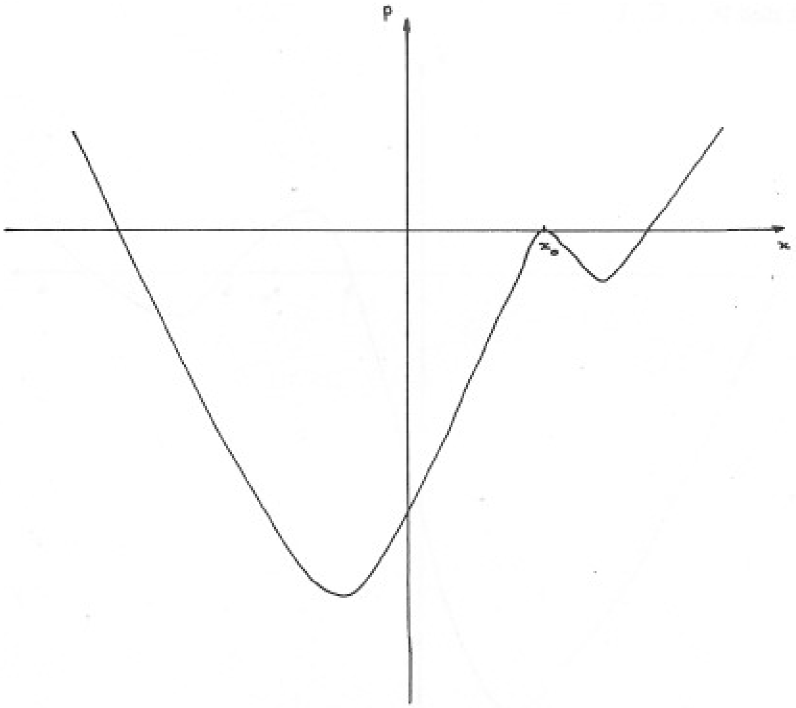

Fig. 2. — Equilibrium position (arbitrary scale).

All of these results are of course valid if our working hypothesis is not violated, that is if the neglected Fourier modes remain negligible. Numerical computations of NLS equation (Yuen and Ferguson [6]) show that this is essentially true if the perturbation wavenumber is in the range:

$$\frac{1}{2}\mu_{cr} < \mu < \mu_{cr},$$

with:

$$\mu_{cr} = 2\sqrt{2}\,a_0 \quad \text{(the critical wavenumber for linear theory).}$$

Finally, let us mention that it would be interesting to compare these results with those obtained by Rabinovich and Fabrikant [7] in their study of stochastic self-modulation. The basic equation used is an extension of NLS equation with dissipative terms; a three mode model is developed too, giving by numerical experiments complex dynamics (strange attractor); equation (6) in [7] is the same as equation (7) here, with a slight change in the

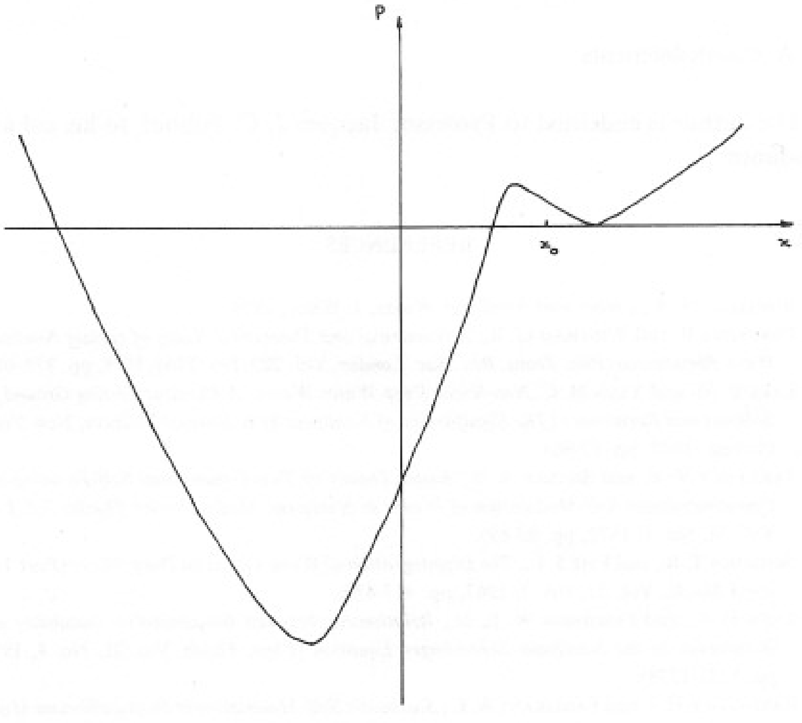

Fig. 3. – Asymptotic case: evanescent perturbations (arbitrary scale).

unknown functions and neglected dissipation. Our study must be considered as a limiting case (no dissipation) of their work, but shows in an analytic way that in this case the system remains quite predictable and deterministic, even when the perturbations have different initial and subsequent values.

3. Conclusion

A Fourier-decomposition approach for NLS equation leads to interesting results. The associated nonlinear set of equations can be solved exactly, confirming the periodic evolution of a wave-train subject to sideband modulations; a complete return to the initial solution can be predicted, within the limitations of the analysis. The Benjamin-Feir "instability" arises only from the linearization procedure, and is in fact the first stage of a recurrence phenomenon, fully nonlinear.

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